Eigenvalue Estimates for Quantum Graphs

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The Laplacian on metric graphs

- Consider a metric graph $\Gamma = (\mathcal{E}(\Gamma), \mathcal{V}(\Gamma))$, $\mathcal{V}(\Gamma) = \{v_i\}_{i \in I}$, $\mathcal{E}(\Gamma) = \{e_j\}_{j \in J}$, where each edge is identified with an interval, $e_j \sim (a_j, b_j)$
- We allow multiple parallel edges between vertices and loops, but our edges will be finite
- Take the Laplacian with "natural" boundary conditions on Γ: models heat diffusion on a graph:
  Laplacian (i.e. second derivative) on each edge-interval;
  continuity plus Kirchhoff condition at the vertices: flow in equals flow out, i.e. the sum of the normal derivatives is zero
- The vertex conditions are generally encoded in the domain of the operator / associated form

The Laplacian on metric graphs

- Formally

$$\begin{aligned} H^1(\Gamma) :=\{u : \Gamma \to \mathbb{R} : u_{|e_j} \in H^1(e_j) \sim H^1(a_j, b_j) \text{ for all edges } e_j \\ \text{and if } e_1 \sim (a_1, b_1) \text{ and } e_2 \sim (a_2, b_2) \text{ share a com-} \\ \text{mon vertex } b_1 \sim a_2, \text{ then } u(b_1) = u(a_2)\} \hookrightarrow C(\Gamma) \end{aligned}$$

- Define a bilinear form $a : H^1(\Gamma) \to \mathbb{R}$ by

$$a(u, v) := \int_\Gamma \nabla u \cdot \nabla v = \sum_j \int_{e_j} u'_{|e_j} v'_{|e_j}, \quad u, v \in H^1(\Gamma)$$

- Call the associated operator in $L^2(\Gamma)$ the Laplacian with natural boundary conditions or “Kirchhoff Laplacian”, $-\Delta_\Gamma$

The eigenvalues of the Laplacian

- Assume Γ is connected and consists of finitely many edges and vertices, and each edge has finite length. Then $-\Delta_\Gamma$ has a sequence of eigenvalues

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty$$

- $\lambda_0 = 0$ with constant functions as eigenfunctions
- Resembles the Neumann Laplacian
  - If Γ consists of a single edge connecting two vertices, it is the Neumann Laplacian on an interval
  - If Γ consists of a single edge connecting the one vertex (i.e. a loop), it is the Laplace-Beltrami operator on a flat circle

Question ("Spectral geometry")

How do the eigenvalues depend on (properties of) Γ?

Spectral geometry on domains/manifolds

- Background: "shape optimisation" on domains or manifolds: which domain optimises an eigenvalue (or combination) among all domains with a given property?
- Classical example: the Theorem of (Rayleigh–) Faber–Krahn: for the Dirichlet Laplacian

$$\begin{aligned} -\Delta u &= \lambda u && \text{in } \Omega \subset \mathbb{R}^d, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned}$$

with eigenvalues $0 < \lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \ldots$,

Theorem

Let $B \subset \mathbb{R}^d$ be a ball with the same volume as Ω. Then $\lambda_1(B) \leq \lambda_1(\Omega)$ with equality iff Ω is (essentially) a ball.

Why? Classical *isoperimetric inequality* plus variational characterisation of λ_1 plus geometry and analysis

Spectral geometry on graphs

We will concentrate (mostly) on λ_1, i.e. the spectral gap

Variational characterisation:

$$\lambda_1(\Gamma) = \inf \left\{ \frac{\|\nabla u\|^2_{L^2(\Gamma)}}{\|u\|^2_{L^2(\Gamma)}} : 0 \neq u \in H^1(\Gamma), \int_\Gamma u = 0 \right\}$$

"Volume" is the total length $L(\Gamma) := \sum_j |e_j| = \sum_j (b_j - a_j)$
Rescaling Γ rescales the eigenvalues accordingly

Theorem (Faber–Krahn-type inequality for graphs; S. Nicaise, 1986; L. Friedlander, 2005; P. Kurasov & S. Naboko, 2013)

$$\lambda_1(\Gamma) \geq \frac{\pi^2}{L^2} = \lambda_1(\textit{line of length } L).$$

Equality holds iff Γ is a line.

In fact $\lambda_k(\Gamma) \geq \frac{\pi^2(k+1)^2}{4L^2}$, $k \geq 1$ (Friedlander)

What properties of Γ should $\lambda_1(\Gamma)$ depend on?

- Length $L(\Gamma)$
- "Surface area of the boundary": Number of vertices $V(\Gamma)$
- Also number of edges $E(\Gamma)$?
- Diameter: $D(\Gamma) = \sup_{x,y\in\Gamma} \mathrm{dist}\,(x, y)$
  Distance is measured along paths within Γ
- The edge connectivity η
- The Betti number $\beta = E - V + 1$
- The Cheeger constant of Γ

  ...

How? Basic variational techniques become much more powerful in one dimension!

"Surgery" on graphs

Recall the variational characterisation

$$\lambda_1(\Gamma) = \inf\left\{ \frac{\|\nabla u\|^2_{L^2(\Gamma)}}{\|u\|^2_{L^2(\Gamma)}} : 0 \neq u \in H^1(\Gamma), \int_\Gamma u = 0 \right\}, \text{ where}$$

$$H^1(\Gamma) = \{u : \Gamma \to \mathbb{R} : u_{|e_j} \in H^1(e_j) \sim H^1(a_j, b_j) \text{ for all edges } e_j$$
$$\text{and if } e_1 \sim (a_1, b_1) \text{ and } e_2 \sim (a_2, b_2) \text{ share}$$
$$\text{a common vertex } b_1 \sim a_2, \text{ then } u(b_1) = u(a_2)\}.$$

- Attaching a *pendant* edge (or graph) to a vertex lowers λ_1 ("monotonicity" with respect to graph inclusion)
- Lengthening a given edge lowers λ_1 (essentially the same)
- Creating a new graph by identifying two vertices raises λ_1
- Adding a new edge between two vertices is a "global" change; the eigenvalue can increase or decrease

Similar principles even hold for the higher eigenvalues λ_k

An upper bound on $\lambda_1(\Gamma)$

Theorem (K.-Kurasov-Malenová-Mugnolo, 2015)

Denote by E the number of edges of Γ. Then

$$\lambda_1(\Gamma) \le \frac{\pi^2 E^2}{L^2}.$$

Equality holds iff Γ is equilateral *and there is an eigenfunction equal to zero on all vertices of Γ.*

- Proof: elementary. Use the surgery principles to reduce to a class of maximisers ("flower graphs", E loops connected to a single vertex) and analyse this class.
- Interesting phenomenon: there are two "types" of maximisers: flower graphs and "pumpkin" (aka "mandarin") graphs

In fact $\lambda_k(\Gamma) \le \frac{\pi^2 E^2 (k+1)^2}{4L^2}$ if Γ is a "tree" (Rohleder, 2016)

Bounds and non-bounds on $\lambda_1(\Gamma)$

- Fix L and V (number of vertices, instead of number of edges). Then $\lambda_1 \to \infty$ is possible.
- Fix E and V. Then $\lambda_1 \to 0$ and $\lambda_1 \to \infty$ are possible. (Rescaling!)

The Cheeger constant

$$h(\Gamma) = \inf_{S \subset \Gamma \text{open}} \frac{\# \partial S}{\min\{|S|, |S^c|\}}.$$

Theorem

$$\frac{h(\Gamma)^2}{4} \leq \lambda_1(\Gamma) \leq \frac{\pi^2 E^2 h(\Gamma)^2}{4}.$$

Optimality of the bounds??

What about diameter D?

Example (K.-Kurasov-Malenová-Mugnolo, 2015)

There exists a sequence of graphs Γ_n ("flower dumbbells") with $D(\Gamma_n) = 1$, $V(\Gamma_n) = 2$ and $\lambda_1(\Gamma_n) \to 0$.

This can be established via a simple test function argument. Much harder (and less obvious) is

Example (K.-Kurasov-Malenová-Mugnolo, 2015)

There exists a sequence of graphs Γ_n ("pumpkin chains") with $D(\Gamma_n) = 1$ and $\lambda_1(\Gamma_n) \to \infty$.

Remark

$\lambda_1(\Gamma_n) \to \infty$ is a "global" property of Γ_n: attach a fixed pendant edge e of length $\ell > 0$ to each Γ_n to form a new graph $\tilde{\Gamma}_n$, then $\lambda_1(\tilde{\Gamma}_n) \le \pi^2/\ell^2$ for all n. (Surgery principle: attaching the pendant graph Γ_n to e can only lower the eigenvalue of e!)

More bounds on $\lambda_1(\Gamma)$?

Theorem (K.-Kurasov-Malenová-Mugnolo, 2015)

If Γ has diameter D, E edges and $V \geq 2$ vertices, then

$$\lambda_1(\Gamma) \leq \frac{\pi^2}{D^2}(V+1)^2$$

and

$$\frac{\pi^2}{D^2 E^2} \leq \lambda_1(\Gamma) \leq \frac{4\pi^2 E^2}{D^2},$$

with equality in the lower bound if Γ is a path and in the upper bound if Γ is a loop.

More bounds on $\lambda_1(\Gamma)$?

Edge connectivity η is the minimum number of "cuts" needed to make Γ disconnected. Rules:

- Vertices cannot be cut;
- Each edge can only be cut once.

Theorem (Band–Lévy '16, Berkolaiko-K.-Kurasov-Mugnolo, '16)

Suppose $\eta(\Gamma) \geq 2$. *Then*

$$\lambda_1(\Gamma) \geq \frac{4\pi^2}{L^2}.$$

(A refinement of Nicaise et al; the proof is a refinement of Friedlander's rearrangement method.) A further refinement:

Theorem (Berkolaiko-K.-Kurasov-Mugnolo, '16)

Suppose $\ell_{\max}$ *denotes the length of the longest edge of* Γ. *Then*

$$\lambda_1(\Gamma) \geq \frac{\pi^2\eta^2}{(L + \ell_{\max}(\eta-2)_+)^2}.$$

Thank you for your attention!